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DESIGNING A RESPONSIVE ENVIRONMENT

SOME DEVICES FOR SELF-INSTRUCTION IN NUMBER

John D Williams
*National Foundation for Educational Research
in England and Wales*

We briefly examine four kinds of device which help the pupil to instruct himself in number structured materials, counting frames, desk calculators and programmed instructional devices Features of design that suit these kinds of device for their job are appraised, and, in some cases, variants are compared for merit

Increasingly, educationists are acknowledging that pupils learn more efficiently in circumstances that have some or all of the following characteristics

(a) the pupil responds actively during the learning process,
(b) he learns as the consequence of a search for information,
(c) he receives immediate 'feedback' relating to the appropriateness of decisions he makes on the basis of his learning,
(d) he controls the rate at which he receives information

The last of these characteristics clearly requires that the pupil should receive the individual 'attention' of his 'instructor', while the other characteristics also require this under many of the learning-arrangements that are conceivable Unfortunately, human instructors are not in plentiful enough supply to provide this indvidual attention Consequently, alternatives must be sought

The obvious alternative is the teaching machine — and, in our final section we indicate something of the role that this can play in instruction in number However, it is not necessary to go this far in seeking non-human instructors, for the behaviour of the number-system can readily be simulated by a variety of devices which will enable the pupil to make exploratory moves, to test himself against 'feedback' and to control his own rate of exposure to information Given that the pupil complies with certain rules, such devices will 'respond' to his learning-requirements much as would an indvidual human instructor

This last sentence may suggest that our message is trivial—for, *given that the pupil complied with certain rules*, just about any set of objects could be used to 'simulate' the number system (I wonder whether 3 plus 5 make 9, so I count my fingers and obtain 'feedback' to the effect that they do not) However, different kinds of object perform this function with differing degrees of effectiveness

In the following pages we shall examine some of the kinds of device that are being used to facilitate self-instruction in number We shall try to bring out the basic possibilities of such devices, and, in some cases, shall show how aspects of the design of a device determine its scope and limitations It is hoped that an analysis of those features will both indicate to the educationist the uses that he might make of available devices and suggest lines along which further devices and learning situations might be designed

STRUCTURED MATERIALS

What are structured materials?

In a general way they can be described as concrete models of the elements of mathematical situations which have been devised so that a pupil can carry out on a concrete level operations that are analogous to mathematical operations (cf 20, 21, 22, 23)

The model form of this material is a set of rods, one to ten units in size, which represents the natural numbers 1—10, and are most readily used for basic number operations However, more elaborate forms, such as the Dienes M A B (4), are more convenient to use in demonstrating powers and positional conventions of number systems (although, with some trouble, the simpler set of one-to-ten unit pieces can be turned to this purpose) Again, forms that are, in a sense *less* elaborate, like the Dienes A E M, can be readily used for demonstrating more general mathematical relations (4), while such basic forms as the Dienes Logical Blocks are best for use in demonstrating mathematical relations of even greater generality, such as those involved in 'sets' (cf 5)

Two merits of structured material

In the teaching of number structured material would appear to have two important advantages In the first place, unlike unit counting materials (such as buttons beads, etc), it presents sets of units in an orderly way which enables the pupil to appreciate the relative size of these sets, and the relative size of sub-sets of units This facility is demonstrated in Figure 1 It is much clearer in the case of the aligned units in Figures *1a* and *1b* that there is an equal number of these units, than it is in the case of the non-aligned units in Figures *1c* and *1d* Likewise it is much clearer in the case of Figure *1b* that the set of units comprises sub-sets of three and four units than it is in the case of Figure *1d*

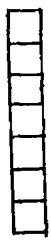

Fig 1a

Fig 1b

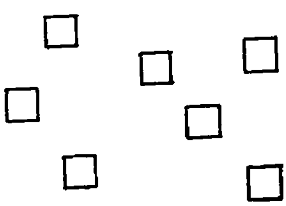

Fig 1c

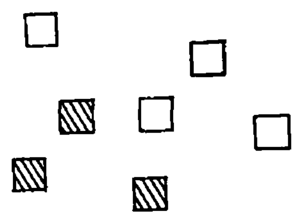

Fig 1d

A second advantage of structured material is that it can be regarded as a 'concrete' set of symbols which actually embody the rules according to which the symbols can be manipulated Because these rules are embodied in the material, the pupil can use the material as a means of discovering them for himself The written symbols that we use for numbers do not in this way embody the rules according to which we should manipulate them There is nothing intrinsic to the symbols '3' and '1' to indicate that when they are added they will be equal to the symbol '4' However, if we took a rod three units in length for our '3' symbol and another rod one unit in length for our '1' symbol, we could combine them to 'make' a rod that in certain respects would be the same as the rod that we use as our '4' symbol

Some basic kinds of use to which structured material can be put

i) Matching a longer piece with combinations of shorter pieces This situation can be used to represent addition ($a + b = c$), subtraction ($c - a = b$), and complementary addition (how much is required to build up to the size of c ?) (cf Figure 2)

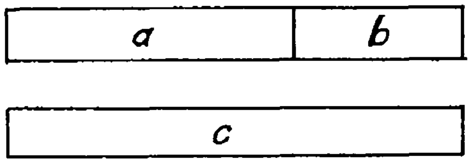

Fig 2

ii) Matching a longer piece with more than one equal shorter pieces in order to illustrate multiplication (3 x a = b), partitive division (b — 3 = a), quotative division (b — a = 3), fractions (if b is unity, a=⅓) or ratios (a b=1 3) (cf Figure 3)

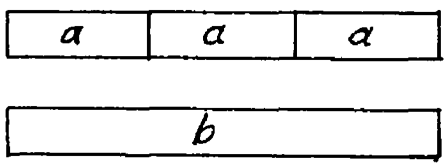

Fig 3

iii) Illustrating powers and place value Dienes M A B materials are particularly suitable for doing this explicitly and consistently and with a variety of bases (cf Figure 4) However, there are ways of using less-

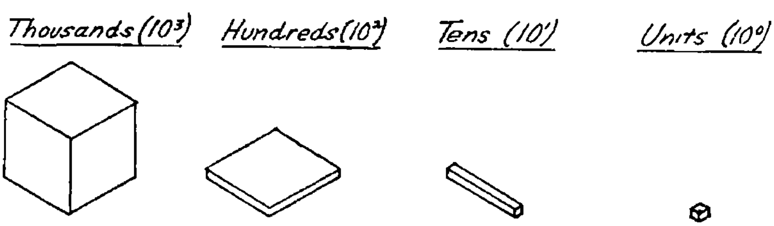

Fig 4

elaborate kinds of material for this purpose (cf Figure 5)

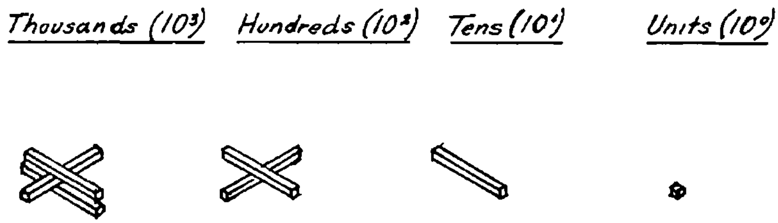

FIG 5

iv) Illustrating basic mathematical relations such as commutation, association and distribution (cf Figure 6)

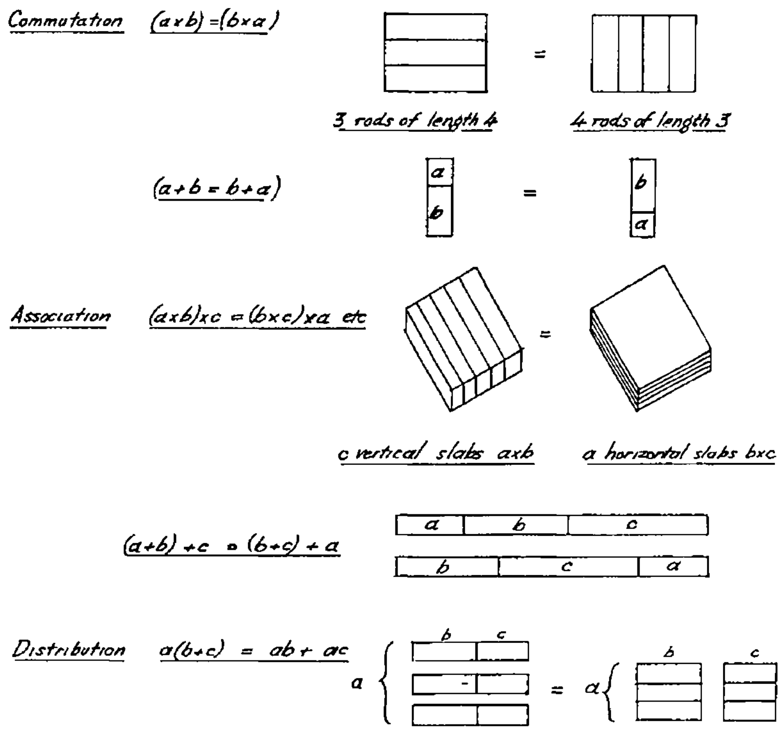

FIG 6

Some designs of models of the first decade

While apparatus that consists of models of the numbers 1—10 conforms in general to the patterns outlined above, there are variations in the design of different kinds of apparatus, and a brief description of some of those kinds that are in fairly common use should provide some idea of the significance of these variations

The Stern apparatus The number-pieces consist of segmented coloured lengths of wood that are multiples of a unit cube The special characteristic of this apparatus is that it includes several containers which serve both to guide the pupil in his manipulations of the apparatus and to provide him with a means of comparing and confirming the structure he has built Worthy of special mention is a board for introducing positional conventions This will hold only nine unit pieces in the units' position Where more than nine unit pieces are to be accommodated these must be exchanged for tens pieces which fit into a tens' position (18)

The Kern apparatus Again, the pieces consist of segmented coloured lengths of wood based on a unit cube, but in this case the number-pieces are coloured in such a way that the pupil is able to see at a glance how a larger piece is made up of smaller components These components are grouped in different ways on the four different sides of the larger piece Thus, in the case of the 5-piece the groupings of units might be as follows first side—one plus four, second side—two plus three third side—two plus one, fourth side—five units Apart from helping to remind the pupil of the composition of particular numbers, this feature of the apparatus enables him to recognise number pieces more readily (10, 11)

Unifix This is similar to the Stern apparatus in that it includes many containers, but is dissimilar in that the units are made of plastic, and may be interlocked so that the pupil can *compose* his own number-pieces The apparatus is thus more flexible number pieces of any size can be made up out of units of various colours One important advantage of this design is that the components of a number can be indicated by colour groupings as they are in the Kern apparatus Another advantage is that subtraction can be represented by actually *removing* part of a number-piece, while addition can be represented by actually *increasing* the size of a number-piece by affixing an extra piece Again, aspects of a rational number can be represented by breaking a larger piece into a number of smaller fractional pieces (19)

The Shaw apparatus Like the Unifix apparatus this consists of interlocking plastic units However, these units are cylindrical and are structured vertically by plugging them either into special bases bearing two rows of ten holes, or into pegboards Instead of containers, cards which fit into slots in the bases and stand vertically behind the constructions, are used for guiding manipulations and measuring structures Apart from the advantages that derive from the use of decomposable number-pieces and which are shared by the Unifix materials, the vertically-structurable Shaw apparatus has two merits It requires less working surface than do those kinds of apparatus that structure horizontally and it possesses a concrete representation of the value of the operator—in the form of the base that supports the structures Holes in a base can be used as a vivid concrete representation of a multiplier or a divisor For example, 3 x 2 can be represented by two units plugged into each of three holes (17, 24, 25)

The Avon apparatus Number pieces are composed of flat square units arranged in double rows On the front of each piece there is a dot per unit, while on the back there is a numeral to indicate the piece's value The dots emphasise the vital configuration formed by each number-piece, which is the more readily recognised since units are arranged in double rows The numerals inscribed on the back of the number-pieces facilitate rapid translation of concrete manipulations into symbolic representations (9)

The Cuisenaire apparatus The number-pieces are systematically coloured but not segmented Pupils learn to refer to the pieces by colour rather than by numerals, and, by certain colour relations, are reminded of certain numerical relations Since the number-pieces are not segmented, the pieces can be used for representing mathematical operations that have not been assigned numerical values (3)

The Colour-Factor apparatus This is similar to the Cuisenaire apparatus, and shares the advantages of this apparatus However, the colouring of the pieces indicate the relationship between factors and their products The number-pieces of this apparatus range from 1—12 units in size, but in other respects deserve to be regarded together with models of the first decade (15)

Some other kinds of structured apparatus

The Montessori apparatus A great variety of apparatus has been devised for use with the Montessori method and, while the name 'Montessori' can hardly be regarded as a newcomer to the teaching of number, much of the more recently developed kinds of structured apparatus can be seen to derive from Montessori originals In addition to number pieces similar to those described above, Motessori used such devices as bead bars counting frames, multiplication and division boards and charts, fraction apparatus, indices and algebra materials and many other kinds of material (14)

The Dienes apparatus The Dienes apparatus has been designed with a view to providing pupils with a variety of models from which to abstract mathematical concepts

The Multibase Arithmetic Blocks consist of models of the decimal and other number systems (4) In the decimal system, units of (10^0) are represented by small cubes, tens (10^1) by 'longs' of 10 structured cubes in a row, hundreds (10^2) by 'flats' of 10 longs arranged to form a square slab and thousands (10^3) by blocks of 10 flats placed on top of one another to form a big cube An illustration of this part of the apparatus is given in Figure 4 Similar models are used for number systems to other bases Thus, the pupil is able to abstract the concept of 'number system' from a variety of different instances An analogous 'triangular trapezoidal' set of models is used as an extra variant

Dienes has also used this principle of variation in devising a set of materials for the teaching of algebra (4) This includes pegboard and pegs, a balance, flat rectangular, triangular and trapezoidal shapes Each mathematical notion is brought home to the pupil by his manipulation of apparatus in more than one of the media

Dienes has also devised a set of 'Logical Blocks' or 'Attribute Blocks' These can be used for a variety of purposes including the teaching of logic (5) They consist of 48 blocks which vary in four respects size, thickness, colour and shape These blocks may be large or small, thick or thin, red, blue or yellow, and square, oblong, triangular or round Dienes has outlined ways in which these blocks can be used for the introduction of sets, and thence for the introduction of number and other kinds of mathematics (6)

The Lowenfeld apparatus Lowenfeld has devised three sets of blocks or *Poleidoblocs* The blocks are in a variety of shapes, all mathematically interrelated, and they can be used both for diagnosing the child's level and style of mathematical thought and for providing the child with an opportunity to discover, by means of play-like manipulations, certain ideas that are basic to mathematical understanding (13)

The Clews Fraction Board Essentially, this consists of several sets of strips of board which are a foot long and various fractions of a foot in width Fractional parts of different sizes are differently coloured, relations among the fractions represented being indicated by relations among the colours used The complete square-foot board is white, the 1/2's, 1/4's, 1/8's, 1/16's sets are each a different reddish colour, the 1/5's and 1/10's are each a different greenish colour and the 1/3's 1/6's and 1/12's are each a different bluish colour Rational number operations are carried out with this apparatus, and their continuity with natural number operations is stressed by the use of the initial letters of the colours of the fractional pieces in an algebraic representation of operations (2)

THE COUNTING FRAME

Many different kinds of counting frame are used in the early teaching of number These range from simple single-line frames which may be used for the introduction of counting, to highly elaborate frames which can be used for a great variety of purposes Probably, the most important role that the counting frame can play in the teaching of number is as an illustration of the conventions of positional notation Several versions of the counting frame are illustrated in Figure 7

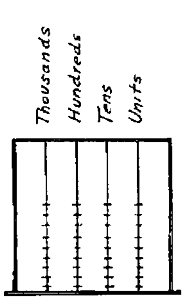

Fig 7a

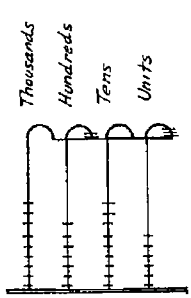

Fig 7b

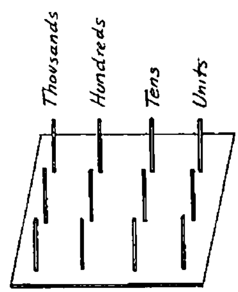

Fig 7c

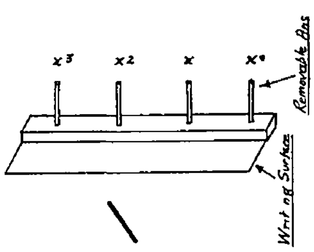

Fig 7d

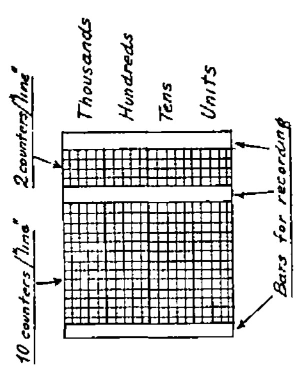

Fig 7e

So that the lines on the counting frame can be seen clearly to represent the columns in the conventional positional arrangements used in recording arithmetical operations on paper, it is desirable that these lines should be vertical, rather than horizontal in arrangement Where this is so, one can have a clear representation of the units', tens' and hundreds' positions as in Figure 7*a*

However, such an arrangement is difficult where frames are upright, for the counters will be displaced by gravity This difficulty can be overcome by the use of a frame in which lines are looped as in Figure 7*b* In this case, where a number is to be registered by moving a counter, the counter is moved over the 'hump'

An alternative to the use of looped lines is illustrated in Figure 7*c* In this case, counters must be placed upon the open-ended lines in order to register values Each line will take just nine counters It can be seen that the example of the open-ended line frame that we have chosen has three rows of lines These can be used to represent the rows of the simple written algorithm This frame has been devised by Sealey (16)

Yet another version of the counting frame—again an open-ended line frame—is prvided by Baas (1) In this case (Figure 7*d*) the lines consist of pins of varying length that can be inserted into a base Thus, lines can be selected so that they will take varying numbers of counters This will permit the adaptation of the frame for use with number systems other than the decimal Attached to the bases is a writing surface which enables the pupil to indicate the order of units for which each line stands

Continuous with these kinds of counting frame is the much more elaborate 'duo-digitus' board of G J Skeats* This is illustrated in Figure 7*e* It includes a central 'counting bar' towards which counters can be displaced in order to register values Counters above this bar signify fives and counters below signify ones In addition to the counting bar, there is a bar at the top of the board and one at the bottom of the board It is possible for the pupil to make symbols with chalk on all three of these bars The top bar is used for indicating the order of units in which the pupil is working, the counting bar is used for recording in numerals the numbers he has registered with the counters, the bottom bar is used for carrying out any interim calculations required It is possible to vary the number of unit counters used in the lower part of the board—at the same time changing the values of those counters in the upper part of the board—for use in working in number systems other than the decimal

In some cases, the difference in significance of the counters on different lines of a counting frame is emphasised by the use of colour For example, counters used as units could be green, those used as tens could be blue, those used as hundreds could be red and so on

*Enquiries about the Skeats Digitus Board should be made of G Skeats, 25 Seapoint Road, Napier, New Zealand

The countıng frame can be seen to afford a more compact model of the number system than does structured apparatus, but ıt can be saıd to be a 'less explıcıt' model, for orders of unıts of dıfferent numerıcal sıze are not represented by materıal of correspondıngly dıfferent physıcal sıze Thıs devıce could well be ıntroduced after work wıth structured apparatus—as a 'semı-concrete brıdge' between a fully concrete representatıon and the use of pen and paper

DESK CALCULATORS

On theoretıcal grounds one mıght expect that desk calculators would be of great use m the classroom Recent experıments wıth these devıces—at both prımary and secondary level—have been very successful and suggest that the effıcıency of the pupıl's self-ınstructıon can be ıncreased consıderably by theır use (7, 12)

Although some sımple models have been marketed at a cheap prıce especıally for use ın the classroom, and although there ıs a great deal of scope for the development of cheap models desıgned specıfically for educatıonal purposes, expense usually prohıbıts the use of more than a few desk caluculators per class However, some of the kınds of use to whıch desk calculators can be put admıt of the allotment of one calculator to a set of pupıls, or of the employment of a lımıted number of calculators for casual use, such as the checkıng of calculatıons, by all pupıls

Pupıls do not usually find much dıfficulty ın dıscoverıng how to operate thıs kınd of machıne, so ınstructıon can well begın wıth an explanatory phase durıng whıch the pupıl dıscovers for hımself how to use the machıne

We can classıfy the uses of these machınes as follows (ı) the dıscovery of mathematıcal relatıons that underlıe arıthmetıc, (ıı) facılıtatıon of the performance of number operatıons ın the course of calculatıons, (ııı) the checkıng of calculatıons

ı) *Mathematıcal relatıons* Basıc mathematıcal prıncıples, whıch underlıe a great number and a great varıety of number operatıons, can be dıscovered the more easıly for the fact that a great number and great varıety of arıthmetıcal operatıons can be carrıed out on a calculator wıth very lıttle dıfficulty In very lıttle tıme, the pupıl can confirm for hımself ın a great varıety of arıthmetıcal sıtuatıons such prıncıples as commutatıon $(a \times b) = (b \times a)$, dıstrıbutıon $a(b + c) = (ab + ac)$ and assocıatıon $(a + b) + c = a + (b + c)$

In many ways the hand-operated calculator makes explıcıt the nature of varıous number operatıons and thereby gıves a concrete demonstratıon of certain mathematical distinctions For example, it is clear that the calculator multiplies by progressive addition and divides by progressive subtraction, so the pupil is constantly reminded of this basic difference between these two kinds of operation Again, in multiplying or dividing, the pupils must first set up the multiplicand or the dividend and then operate upon it physically by turning the handle of the machine, he is thus brought to realise the distinction between operand and operator Again, that division is the reverse of multiplication is clearly brought home to the pupil by the fact that he must turn the handle in one direction for multiplication and in the opposite direction for division Finally, many aspects of decimal notation are emphasised by the relationship between moving the machine's carriage and operating by different powers of ten

Characteristics of number patterns and series can be discovered with very little difficulty, as can many other characteristics of the behaviour of numbers With the facility that the machine provides for accurate and easy calculation, playful exploration of the possibilities of numbers is likely to be a much more appetising prospect for the pupil

ii) *Facility in calculation* Inaccuracy in calculation is a main source of failure in arithmetic and a discouragement from the exploration of its mathematical properties Characteristics of the child that are relatively irrelevant to his capacity for mathematical insight may be responsible for this inaccuracy—for example, the inability to sustain attention throughout a long and tedious written calculation, or the failure to master some of the multiplication tables Both of these kinds of inaccuracy can be obviated by the use of a machine, which enables the pupil to perform calculations speedily and without much reliance upon multiplication tables Thus, many pupils are no longer prevented by computational inaccuracy from successfully engaging in mathematically useful activities

Problems of a more meaningful kind can be tackled with the help of a desk calculator Because the real world tends to be arithmetically untidy, problem arithmetic as traditionally carried out consists of rather stereotyped and unrealistic examples contrived in such a way that the arithmetic that they involve is simple enough for the pupil to tackle However, where complex arithmetical situations can be coped with easily by means of a desk calculator, the teacher is able to avail himself of a much wider range of problems, and, in particular, of problems that are taken from the arithmetically-untidy realm of real existence Again, where problems no longer need to be of the kind that 'work out', the pupil can be given a great deal more freedom to construct his own

Finally, where the pupil's attention is not entirely taken up with the mechanics of computational procedures, it can be concentrated more upon the basic structure of a problem situation Instead of proceeding blindly and slowly through a confusing mass of calculations, pupils will be able to give more attention to the general features of the calculative activity, and thus appreciate its mathematical relevance

iii) *Checking* Calculators facilitate two kinds of self-checking first, where it is not convenient for the pupil to use the calculator in carrying out his computations (for example, where only one machine is available for several pupils), written calculations can be checked against the machine, which constitues an independent and reliable means of assessment Second, calculations carried out on the machine can likewise be checked—a particularly convincing and illuminating way of carrying out such checks is to reverse the procedure used in performing the original calculation

As we shall emphasise in the section on programmed instruction, this facility for self-correction is of great importance, for it helps the pupil to drop erroneous behaviours before they become habits, and it motivates performance

PROGRAMMED INSTRUCTIONAL DEVICES

In the case of mathematics, more than in that of most subjects, many devices that approach the 'teaching machine' in important respects have been used for some time at an experimental level Mathematics lends itself to programmed presentation, and has been the area in which most programming has been taking place (cf 8)

In this section we shall point out some of the main characteristics and advantages of programmed instruction then point out why mathematics in particular can benefit *from* this kind of presentation and how it lends iteslf *to* this kind of presentation It will be seen that in making these points we shall also, to a considerable extent, be justifying the policy of providing the pupil with the opportunity to control his own instruction

The basic characteristics and merits of programmed instruction

Essentially, this method involves breaking up a course of instruction into very small steps which can be presented to the pupil in a pre-arranged order—or, in the case of 'branching programmes' as a choice among several pre-arranged orders The programme thus created is usually subjected to 'internal-validational' procedures which establish its weak points so that they may be corrected It is partly in this detailed analysis and validation of a learnt sequence that the merit of programmed instruction lies

In most cases, the actual learning situations that obtain in this kind of presentation have three important characteristics In the first place, the pupil is presented with a stimulus (for example, 2 x 3 = ?) which might give him information, or demand of him a response or do both of these simultaneously Secondly, at every step in the programme, the pupil is required to utilise the provided information or his previous learning in making some kind of response This ensures both that the pupil plays an active role in learning and that he proceeds through the programme in the order for which it has been designed Thirdly, immediately a response has been made, the pupil is informed of its appropriateness In many cases, this information consists of the correct answer to any question that the pupil has been asked Advantages of this immediate feed-back have been mentioned in the preceding section it enables the pupil to recognise at an early stage any errors and provides him with motivation

Within this general framework there are many variations, both in the design of programmes and in the kinds of device used in presenting them Devices take many forms simple sets of cards that a teacher has prepared himself, programmed text-books simple mechanical devices (for the teaching and practising of elementary arithmetic such mechanical devices are particularly readily constructed) more expensive electronic devices, versatile computer-based machines that adjust to the pupil's requirements along many dimensions

The importance of programming mathematics

It is quite widely acknowledged that mathematical techniques and insights tend to be 'cumulative' in nature—that is the possibility of later learning of skills and later acquisition of information and concepts is to a great extent dependent upon earlier learning Pupils who have failed to learn basic techniques and concepts may not stand a chance of learning those techniques and concepts that are dependent upon these Programmed instruction is able to combat this source of difficulty in learning mathematics in three important ways First of all, since programmes must be planned ahead, the subject-matter is analysed in great detail, and logical interdependencies of parts are acknowledged in the sequencing of material Therefore, it is unlikely that improper sequencing of learning experiences will be responsible for the pupil's failure to acquire those learnings that are necessary for later learnings Secondly, since a programmed instructional device can deal with a particular pupil's requirements, each pupil can proceed at his own pace, and will not need to sacrifice mastery of any part of the course in order to keep up with a teacher-paced class Where the pupil's learning has been interrupted, by absence from school, for example, he will be able to resume his work at the point at which he left it and thus will have the opportunity of mastering basic steps without disturbing the organisation of the rest of the class And thirdly, as we have already pointed out, the amenity of immediate 'feed-back' makes it possible for the pupil to discover and correct errors as soon as they are made, thus ensuring that a basic misconception will not interfere with later learning

How mathematics lends itself to programming

It is not an accident that more headway has been made in the programming of mathematics than in that of other subject-matter This is partly because of the great pressure to improve mathematics teaching, but partly, also, because mathematics particularly lends itself to programming

For one thing, mathematics is particularly easy, in comparison with many other subjects, to analyse into parts and sequences Its parts are interrelated in accordance with a relatively clear logical structure, and it is possible to find out which parts it is *logically* necessary to master before other parts can be mastered It is not claimed that this amenability to logical analysis completely solves the problem of dividing and sequencing mathematics for characteristics of the learner must be taken into consideration as well as characteristics of the subject-matter However, the logical structure of mathematics does give us *some* basis for a procedure for organising the presentation of mathematical content

A further reason for the development of programming in mathematics is that much of mathematics—especially calculative procedures—lends itself admirably to a method of presentation in which the pupil is required to learn by responding actively This is because much of mathematics requires, for its learning, the mastery of clearly defined skills of calculation

Again, since mathematics is a fairly rigorous discipline, it is possible to say with a high degree of confidence whether a move in the 'mathematical game' is right or wrong Because of this, it is possible to pre-arrange learning situations in which the pupil can be apprised of the correctness of the 'moves' that he has made It is likely to be argued by modern-mathematical educationists that the correctness of a mathematical move is by no means always clear—especially where the criterion of correctness is to be the efficiency of learning—but, given this, there are senses in which it could be said to be more clear than it would be in the case of other subject-matters

It is frequently possible to build an extremely flexible machine which will act as an analogue to certain limited parts of mathematics For example, a calculating machine can be said to act as an analogue to a great deal of the system of arithmetic Where the construction of such analogues is possible, it may be possible also to use simple keyboard devices for registering the learner's response

Finally, since mathematics uses its own symbolism, and since this symbolism is much simpler in many ways than the symbolism used in natural languages, the reading demands on the learner can be reduced to a very low level, and hence it will be possible to provide him with intelligible stimuli in a programme without additional explanation from a teacher, even when his reading age is relatively low For example, many poor readers are able to master the symbolic conventions of arithmetic and carry out calculations despite their reading deficiencies

Although programmed instruction would seem for several reasons to be a suitable technique for the teaching of mathematics and, although, in principle, programmes of great flexibility and resource can be constructed, it is perhaps necessary to urge that care should be taken lest the flexibility of mathematical learning situations should be reduced by the detailed pre-planning of such situations

CONCLUSION

It is hoped that the above accounts of structured materials, counting frames desk calculators, and programmed instruction will provide the educationist with some idea of how such devices may be used in the design of learning situations that enable the pupil to instruct himself in number All these devices are 'responsive', in that they not only provide the pupil with an opportunity to exercise what he has learnt, but also give him intelligible 'feed-back' relating to the appropriateness of his endeavours Thus they assume some of the teacher's burden, while facilitating self-paced discovery learning on the part of the pupil

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